

A remark on primitive cycles and Fourier-Radon transform

A. Beilinson

The aim of this note is to point out that Brylinski's Radon transform [B] is a natural instrument for the Green-Griffiths approach to Hodge conjecture [GG], [BFNP]. In particular, some principal results of [BFNP] follow from the general fact that Radon transform preserves primitive cohomology (while reversing its grading). As was noticed by Drinfeld, this assertion is immediate from the basic Fourier transform functoriality [L].¹

This note originates from a talk given at a student Hodge theory seminar. I am grateful to V. Drinfeld for his enlightening comment, to D. Kazhdan for a discussion, and to M. Kerr and G. Pearlstein for an exchange of letters.

1. A reformulation of the Hodge conjecture. For a compact complex algebraic variety X let $N_i H^*(X, \mathbb{Q})$ be the niveau filtration on its cohomology (it is Poincaré dual to more commonly used coniveau filtration; conjecturally, the two filtrations are complementary). Thus $N_1 H^*(X, \mathbb{Q})$ is the intersection of kernels of all restriction maps $H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$, where $Y \neq X$ is a closed algebraic subvariety of X . According to Totaro and Thomas, see [BFNP] th. 6.5, the Hodge conjecture amounts to the next assertion: For every projective smooth X of dimension $2n$ the subspace of Hodge (n, n) -classes in $H^{2n}(X, \mathbb{Q})$ has zero intersection with $N_1 H^{2n}(X, \mathbb{Q})$. Of course, it suffices to consider the subspace of primitive Hodge classes. Thus every description of $N_1 H^{2n}(X, \mathbb{Q})^{\text{prim}}$ provides a reformulation of the Hodge conjecture. The articles [GG] and [BFNP] provide one such description; we present it in the last line of the note.

REMARK. As was pointed out by the referee, Kerr and Pearlstein can treat similarly Grothendieck's generalized Hodge conjecture.

QUESTION. For γ in a given term of coniveau filtration, what can one say about simplest possible singularities of Y with $\gamma|_Y \neq 0$? (E.g., by Thomas, for algebraic γ , i.e., for γ in the deepest term of coniveau filtration, the singularities are ODP.)

2. Radon transform ([B]). We play with complex algebraic varieties and \mathbb{Q} -sheaves. An arbitrary ground field and \mathbb{Q}_ℓ -sheaves will do as well.

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¹My initial argument was less elegant (it used relative Lefschetz decomposition).

For an algebraic variety Z , we denote by $D(Z)$ the derived category of bounded constructible \mathbb{Q} -complexes on Z ; let $\mathcal{M}(Z) \subset D(Z)$ be the category of perverse sheaves on Z , ${}^pH : D(Z) \rightarrow \mathcal{M}(Z)$ the cohomology functor ([BBD]). For smooth Z let $\mathcal{M}^{\text{sm}}(Z) \subset \mathcal{M}(Z)$ be the Serre subcategory of smooth perverse sheaves (i.e., local systems); it generates the thick subcategory $D^{\text{sm}}(Z) \subset D(Z)$ of complexes with smooth cohomology. The Verdier quotient $\bar{D}(Z) := D(Z)/D^{\text{sm}}(Z)$ is a t-category with heart $\bar{\mathcal{M}}(Z) := \mathcal{M}(Z)/\mathcal{M}^{\text{sm}}(Z)$. The latter is an Artinian \mathbb{Q} -category; the projection $\mathcal{M}(Z) \rightarrow \bar{\mathcal{M}}(Z)$ identifies the subcategory of non-smooth irreducible perverse sheaves on Z with that of irreducible objects in $\bar{\mathcal{M}}(Z)$.

Let V be a vector space of dimension $n \geq 2$, V^\vee its dual. Let $\mathbb{P}, \mathbb{P}^\vee$ be the corresponding projective spaces, $i : T \hookrightarrow \mathbb{P} \times \mathbb{P}^\vee$ be the incidence correspondence. Let p, p^\vee be the projections $\mathbb{P} \times \mathbb{P}^\vee \rightrightarrows \mathbb{P}, \mathbb{P}^\vee$, and $p_{(T)}, p_{(T)}^\vee$ be their restrictions to T . The Radon transform functor $\mathcal{R} : D(\mathbb{P}) \rightarrow D(\mathbb{P}^\vee)$ is $\mathcal{R}(M) := p_{(T)!}^\vee p_{(T)}^* M[n-2]$. Interchanging \mathbb{P} and \mathbb{P}^\vee , we get $\mathcal{R}^\vee : D(\mathbb{P}^\vee) \rightarrow D(\mathbb{P})$, etc. Notice that \mathcal{R} sends $D^{\text{sm}}(\mathbb{P})$ to $D^{\text{sm}}(\mathbb{P}^\vee)$, so we have $\bar{\mathcal{R}} : \bar{D}(\mathbb{P}) \rightarrow \bar{D}(\mathbb{P}^\vee)$.

THEOREM ([B] 3.1). *The compositions $\bar{\mathcal{R}}\bar{\mathcal{R}}^\vee, \bar{\mathcal{R}}^\vee\bar{\mathcal{R}}$ are Tate twist functors $M \mapsto M(2-n)$. The functors $\bar{\mathcal{R}}, \bar{\mathcal{R}}^\vee$ are t-exact, hence they yield equivalences of the abelian categories $\bar{\mathcal{M}}(\mathbb{P}) \simeq \bar{\mathcal{M}}(\mathbb{P}^\vee)$. \square*

3. Fourier transform ([B], [L]). The formalism of constructible sheaves extends to algebraic stacks of finite type ([LMB], [LO]). The group \mathbb{G}_m acts on any vector space by homotheties. Consider the quotient stacks $\mathcal{V} := V/\mathbb{G}_m, \mathcal{V}^\vee := V^\vee/\mathbb{G}_m, \mathcal{A}^1 := \mathbb{A}^1/\mathbb{G}_m$. The open embedding $j_V : V^\circ := V \setminus \{0\} \hookrightarrow V$ yields one $j_V : \mathbb{P} \hookrightarrow \mathcal{V}$, etc. The canonical pairing map $\mu : V \times V^\vee \rightarrow \mathbb{A}^1$ yields $\mu : \mathcal{V} \times \mathcal{V}^\vee \rightarrow \mathcal{A}^1$. Let $pr, pr^\vee : \mathcal{V} \times \mathcal{V}^\vee \rightrightarrows \mathcal{V}, \mathcal{V}^\vee$ be the projections. One has the (homogenous) Fourier transform $\mathcal{F} : D(\mathcal{V}) \rightarrow D(\mathcal{V}^\vee)$, $\mathcal{F}(N) := pr_!^\vee(pr^* N \otimes \mu^* j_{\mathcal{A}^1}^* \mathbb{Q})[n-1]$, see [L] 1.5, 1.9. Interchanging \mathcal{V} and \mathcal{V}^\vee , we get $\mathcal{F}^\vee : D(\mathcal{V}^\vee) \rightarrow D(\mathcal{V})$.

THEOREM ([L] 3.1, 4.2). *The compositions $\mathcal{F}\mathcal{F}^\vee, \mathcal{F}^\vee\mathcal{F}$ are Tate twist functors $N \mapsto N(-n)$. The functors $\mathcal{F}, \mathcal{F}^\vee$ are t-exact, hence they yield equivalences of the abelian categories $\mathcal{M}(\mathcal{V}) \simeq \mathcal{M}(\mathcal{V}^\vee)$. \square*

Consider the closed embeddings $i_V : \{0\} \hookrightarrow V, i_V : B_{\mathbb{G}_m} = \{0\}/\mathbb{G}_m \hookrightarrow \mathcal{V}$, etc. The projection $j_{\mathcal{A}^1}^* \mathbb{Q} \rightarrow i_{\mathcal{A}^1}^* \mathbb{Q}(-1)[-1]$ yields a natural morphism

$$(1) \quad j_{\mathcal{V}^\vee}^* \mathcal{F} j_{\mathcal{V}!} \rightarrow \mathcal{R}(-1),$$

which becomes an isomorphism $j_{\mathcal{V}^\vee}^* \mathcal{F} j_{\mathcal{V}!} \xrightarrow{\sim} \bar{\mathcal{R}}(-1)$ in $\bar{D}(\mathbb{P}^\vee)$ (see [L] 1.6). By [L] 1.8, one has a natural identification

$$(2) \quad \mathcal{F} i_{\mathcal{V}!} \xrightarrow{\sim} \pi_{B_{\mathbb{G}_m}}^* [n],$$

where $\pi_{B_{\mathbb{G}_m}}$ is the projection $\mathcal{V}^\vee \rightarrow B_{\mathbb{G}_m}$. Notice that $\pi_{B_{\mathbb{G}_m}}^* \xrightarrow{\sim} i_{\mathcal{V}^\vee}^*$, so $\pi_{B_{\mathbb{G}_m}}^*$ is left adjoint to $i_{\mathcal{V}^\vee}^*$. Passing in (2) to the right adjoint functors, we get

$$(3) \quad i_{\mathcal{V}}^! [n] \xrightarrow{\sim} i_{\mathcal{V}^\vee}^* \mathcal{F}.$$

Remark. Other settings for Fourier transform of constructible sheaves can be also used towards our aim (these are monodromic Fourier transform that identifies

the subcategories of complexes with monodromic cohomology in $D(V)$ and $D(V^\vee)$,² and, for \mathcal{D} -modules or for ℓ -adic sheaves in finite characteristic, the full Fourier transform that identifies $D(V)$ with $D(V^\vee)$.

4. Primitive cycles. Let M be a non-constant irreducible perverse sheaf on \mathbb{P} . By the theorem in 2, $\bar{\mathcal{R}}(M)$ is an irreducible object of $\bar{\mathcal{M}}(\mathbb{P}^\vee)$; let M^\vee be the corresponding non-constant irreducible perverse sheaf on \mathbb{P}^\vee . Let $c \in H^2(\mathbb{P}, \mathbb{Q}(1))$ be the class of a hyperplane section. We have the primitive decomposition³

$$(4) \quad \bigoplus_{j \geq \max\{a/2, 0\}} H^{a-2j}(\mathbb{P}, M(-j))^{\text{prim}} \xrightarrow{\sim} H^a(\mathbb{P}, M),$$

where $H^{-i}(\mathbb{P}, M)^{\text{prim}} := \text{Ker}(c^i : H^{-i}(\mathbb{P}, M) \rightarrow H^i(\mathbb{P}, M)(i))$, $i \geq 0$, the j -component of $\xrightarrow{\sim}$ is multiplication by c^j . Set $H^a(\mathbb{P}, M)^{\text{coprim}} := \text{Ker}(c : H^a(\mathbb{P}, M) \rightarrow H^{a+2}(\mathbb{P}, M)(1))$, $a \geq 0$, which equals component $j = 2a$ of (4). Ditto for M^\vee .

THEOREM. *One has canonical identifications*

$$(5) \quad H^a(\mathbb{P}, M)^{\text{coprim}} \xrightarrow{\sim} H^{a+2-n}(\mathbb{P}^\vee, M^\vee)^{\text{prim}}.$$

PROOF. The intermediate extension functor $j_{V!} : \mathcal{M}(\mathbb{P}) \rightarrow \mathcal{M}(\mathcal{V})$, $j_{V!}(M) := \text{Im}({}^p H^0 j_{V!}(M) \rightarrow {}^p H^0 j_{V*}(M))$ identifies the category of irreducible perverse sheaves on \mathbb{P} with that of those irreducible perverse sheaves on \mathcal{V} which are not supported on $\mathcal{V} \setminus \mathbb{P} = \{0\}/\mathbb{G}_m$. Since \mathcal{F} sends sheaves supported on $\mathcal{V} \setminus \mathbb{P}$ to constant sheaves and ${}^p H^0 j_{V!}(M) = j_{V!}(M)$, we see that (1) yields $j_{V^\vee}^* \mathcal{F} j_{V!}(M) \xrightarrow{\sim} M^\vee(-1)$, hence $j_{V^\vee!}(M^\vee) = \mathcal{F} j_{V!}(M)(1)$. Applying (3), we get $i_V^! j_{V!}(M)(1) = i_{V^\vee}^* j_{V^\vee!}(M^\vee)[-n]$. Pulling it back by the smooth projections $\pi_V : V \rightarrow \mathcal{V}$, $\pi_{\mathbb{P}} : V^\circ \rightarrow \mathbb{P}$ of relative dimension one, we get a canonical isomorphism

$$(6) \quad i_V^! j_{V!}(M^b)(1) \xrightarrow{\sim} i_{V^\vee}^* j_{V^\vee!}(M^{\vee b})[-n],$$

where $M^b := \pi_{\mathbb{P}}^* M[1]$, $M^{\vee b} := \pi_{\mathbb{P}^\vee}^* M^\vee[1]$ are irreducible perverse sheaves on V° , $V^{\vee \circ}$. Since i_V^* is right t-exact and $j_{V!}(M^b)$ is irreducible, the complex $i_V^* j_{V!}(M^b)$ is acyclic in degrees ≥ 0 ; dually, $i_V^! j_{V!}(M^b)$ is acyclic in degrees ≤ 0 . We get (5) combining (6) with the next (well-known) lemma:

LEMMA. *There are canonical identifications $H^a i_V^* j_{V!}(M^b) \xrightarrow{\sim} H^{a+1}(\mathbb{P}, M)^{\text{prim}}$, $H^a i_V^! j_{V!}(M) \xrightarrow{\sim} H^{a-1}(\mathbb{P}, M(-1))^{\text{coprim}}$.*

PROOF OF LEMMA. The canonical exact triangle $i_V^! j_{V!}(M^b) \rightarrow i_V^* j_{V!}(M^b) \rightarrow i_V^* j_{V*}(M^b)$ and the above acyclicity remark imply that

$$(7) \quad i_V^! j_{V!}(M^b)[1] = \tau_{\geq 0} i_V^* j_{V*}(M^b), \quad i_V^* j_{V!}(M^b) = \tau_{< 0} i_V^* j_{V*}(M^b).$$

Now $i_V^* j_{V*}(M^b) \xrightarrow{\sim} R\Gamma(V^\circ, M^b) \xrightarrow{\sim} R\Gamma(\mathbb{P}, M \otimes \pi_{\mathbb{P}*} \mathbb{Q}_{V \setminus \{0\}})[1]$, the first isomorphism comes since M^b is \mathbb{G}_m -equivariant, the second is the projection formula. Thus the evident exact triangle $\mathbb{Q}_{\mathbb{P}} \rightarrow \pi_{\mathbb{P}*} \mathbb{Q}_{V \setminus \{0\}} \rightarrow \mathbb{Q}(-1)_{\mathbb{P}}[-1]$, its boundary map is c , yields isomorphism $i_V^* j_{V*}(M^b) \xrightarrow{\sim} \text{Cone}(R\Gamma(\mathbb{P}, M(-1))[-1] \xrightarrow{c} R\Gamma(\mathbb{P}, M)[1])$. By (7) and (4), it provides the identifications of the lemma, q.e.d. \square

²Monodromic Fourier transform is the functor $N \mapsto \text{holim}_a pr_1^\vee(pr^* N \otimes \mu^* j_{A^1*} \mathcal{L}_a)[n+1]$, where $\dots \rightarrow \mathcal{L}_2 \rightarrow \mathcal{L}_1$ are local systems on $A^1 \setminus \{0\}$ with unipotent Jordan block monodromy, $\text{rk } \mathcal{L}_a = a$. For the analytic version, see [B] §6.

³For an arbitrary irreducible F this was proven in [D] (via [BK] or [G]) and [M].

REMARK. Only the case $a = 0$ is needed for the aims of [BFNP].

5. A description of N_1 ([GG], [BFNP]). Let X be an irreducible projective variety, F be an irreducible perverse sheaf on X whose support equals X (the case we need is $F = \mathbb{Q}_X[\dim X]$), \mathcal{L} be a very ample sheaf on X . Let us describe the subspace $N_1 H^0(X, F)^{\text{prim}}$ of $H^0(X, F)^{\text{prim}}$ (which is the intersection of kernels of all restriction maps to $H^0(Y, F|_Y)$, Y is a closed proper subspace of X).

We have the embedding $i_{\mathcal{L}} : X \hookrightarrow \mathbb{P} = \mathbb{P}_{\mathcal{L}}$, $n = \dim H^0(X, \mathcal{L})$, that corresponds to \mathcal{L} ; we assume that $X \neq \mathbb{P}$, so $M := i_{\mathcal{L}*} F$ is non-constant. Consider identification $\alpha : H^0(X, F)^{\text{prim}} \xrightarrow{\sim} H^{2-n}(\mathbb{P}^{\vee}, M^{\vee})$ defined as the composition $H^0(\mathbb{P}, M)^{\text{prim}} = H^0(\mathbb{P}, M)^{\text{coprim}} \xrightarrow{\sim} H^{2-n}(\mathbb{P}^{\vee}, M^{\vee})^{\text{prim}} = H^{2-n}(\mathbb{P}^{\vee}, M^{\vee})$ where $\xrightarrow{\sim}$ is the isomorphism from (5).

For a constructible complex G denote by $\mathcal{H}^* G$, τ_{\geq} its usual (not perverse) cohomology sheaves and the canonical truncation. The projection $M^{\vee} \rightarrow \tau_{\geq 2-n} M^{\vee}$ yields the map $H^{2-n}(\mathbb{P}^{\vee}, M^{\vee}) \rightarrow H^0(\mathbb{P}^{\vee}, \mathcal{H}^{2-n} M^{\vee})$. Let $K_{\mathcal{L}} \subset H^0(X, F)^{\text{prim}}$ be the α -preimage of its kernel. It coincides with the kernel of the composition $H^0(X, F)^{\text{prim}} \hookrightarrow H^0(X, F) \xrightarrow{p_{(T)}^*} H^{2-n}(\mathbb{P}^{\vee}, \mathcal{R}(M)) \rightarrow H^0(\mathbb{P}^{\vee}, \mathcal{H}^{2-n} \mathcal{R}(M))$, which assigns to a primitive cycle the display of its images in $H^0(Y, F|_Y)$ for all hyperplane sections Y . (Indeed, by the decomposition theorem, $\mathcal{R}(M)$ is the direct sum of M^{\vee} and constant sheaves, so the kernel of a projection $\mathcal{H}^{2-n} \mathcal{R}(M) \rightarrow \mathcal{H}^{2-n}(M^{\vee})$ is a constant sheaf, and primitive cycles restrict to 0 on a general hyperplane section.) Therefore $K_{\mathcal{L}}$ consists of all primitive cycles whose restriction to each hyperplane section is 0.

Clearly $K_{\mathcal{L}} \subset K_{\mathcal{L}^{\otimes 2}} \subset \dots$. Since every closed subscheme $Y \subset X$, $Y \neq X$, lies on a hypersurface of sufficiently high degree, we see that $N_1 H^0(X, F)^{\text{prim}}$ equals $K_{\mathcal{L}^{\otimes n}}$ for $n \gg 0$.

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